Mathematics 222B Lecture 25 Notes

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1 The de-Giorgi-Nash-Moser Theorem

1.1 How the theorem answers Hilbert's 19th problem

Today, we will be concluding our discussion of the solution to Hilbert's 19th problem, which was posed in 1900. Here is the problem:

Problem 1.1. *Assume $L = L(p)$ is convex and analytic. Prove that minimizers of $\mathcal{F}[u] = \int_U L(Du)\,dx$ are analytic.*

The original problem was stated for $d = 2$ and was solved by Morrey (at Berkeley). Later, Nash solved the problem for $d \geq 3$, but it turns out that de Giorgi solved the problem (with a slightly different theorem) a few years earlier; so both get the credit. Later, Moser simplified the theory and proved a number of other theorems along the way. So this is generally referred to as de Giorgi-Nash-Moser theory.

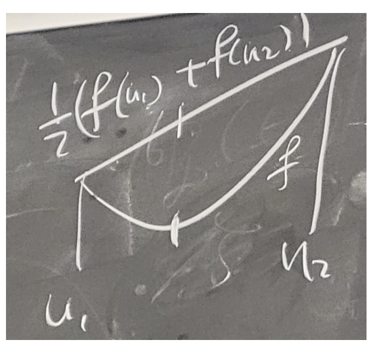
Today, we will be proving the following theorem

Theorem 1.1 (de Giorgi-Nash-Moser)**.** *Assume $L \in C^\infty(\mathbb{R}^d)$ and L is uniformly convex, i.e.*

$$\lambda|\xi|^2 \leq \partial_{p_j}\partial_{p_k} L \xi_j \xi_k \leq \Lambda|\xi|^2.$$

Then for all $V \subseteq\subseteq U$, the minimizer $u \in C^\infty(V)$.

Remark 1.1. With uniform convexity, the uniqueness of the minimizer follows.



Convexity of a function always tells you that

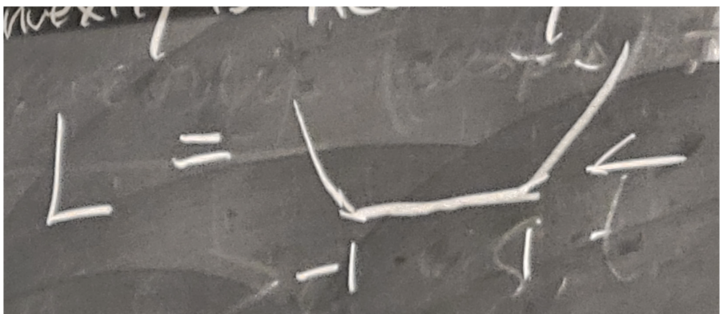
$$f\left(\frac{u_1+u_2}{2}\right) \leq \frac{f(u_1)+f(u_2)}{2},$$

and strict convexity means that equality holds iff $u_1 = u_2$. So let u_1, u_2 be minimizers for L. Then

$$\int_U L\left(D\left(\frac{u_1+u_2}{2}\right)\right) \leq \int_U \frac{1}{2}L(Du_1) + \frac{1}{2}L(Du_2).$$

This means that $\frac{u_1+u_2}{2}$ is also a minimizer, so strict convexity gives $Du_1 = \frac{Du_1+Du_2}{u_2}$. So $Du_1 = Du_2$ in ∂U, and since u_1, u_2 agree on the boudnary, we get $u_1 = u_2$ in U.

Remark 1.2. Strict convexity is necessary for the theorem. Consider the following example in $d = 1$:



Then $|x|$ would be a minimizer, so

$$L(D|x|) = \int L_{\min}\, dx = \ell,$$

but $|x|$ is only Lipschitz.

1.2 Reduction to $u \in C^{1,\alpha}(V)$

Now we will prove a key reduction to $u \in C^{1,\alpha}(V)$. The keyword here is "standard elliptic theory," and in particular L^2 and Schauder theory. The minimizer will satisfy the Euler-Lagrange equation

$$\partial_{x^j}(\partial_{p_j} L(Du)) = 0.$$

The minimizer $u \in H^1(U)$ solves this equation in the weak sense. Let us differentiate this once more. Letting $w_i = \partial_i u$, we will have that each w_i solves the linearized Euler-Lagrange equation

$$\partial_j\left(\frac{\partial^2}{\partial p_j \partial p_k} L\bigg|_{p=Du} \partial_k w_i\right) = 0.$$

The term $\frac{\partial^2}{\partial p_j \partial p_k} L|_{p=Du}$ is uniformly elliptic (i.e. $\lambda|\xi|^2 \leq a^{j,k}\xi_j\xi_k$ with $|a| \leq \Lambda$) and in L^∞. This tells us that $w_i \in H^1(U)$, which follows from standard L^2-elliptic regularity theory (see Evans section 8.3 for details).

But still, all we know is that $a^{j,k} \in L^\infty$. What do we need? All we need is to show that $a^{j,k} \in C^{0,\alpha}$ for some $\alpha > 0$. Remember our equation is

$$\partial_j(a^{j,k}\partial_k w) = 0.$$

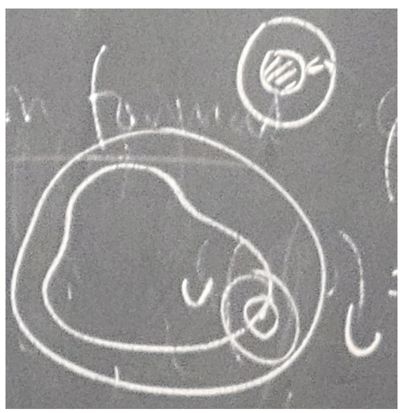
If $a^{j,k} \in C^{0,\alpha}$, then by Schauder theory, $w \in C^{1.\alpha}$. Then we have that $u \in C^{2,\alpha}$, so $a^{j,k} \in C^{1,\alpha}$. Then we get $w \in C^{2,\alpha}$, and we repeat. This is called an (elliptic) bootstrap argument.

The heart of the de Giorgi-Nash-Moser theory is to show that $a^{j,k} \in C^{0,\alpha}$ for some $\alpha > 0$. Now it suffices to show the following theorem.

Theorem 1.2. *Let $w \in H^1(B_1)$ be a solution to $Pw = -\partial_j(a^{j,k}\partial_k w) = 0$. Assume that $a \in L^\infty$ and $\lambda|\xi|^2 \leq a^{j,k}(x)\xi_j\xi_k \leq \Lambda|\xi|^2$. Then*

$$\|w\|_{C^{0,\alpha}(B_{1/2})} \lesssim_{d,\lambda,\Lambda} \|w\|_{L^2(B_1)}.$$

Here we only need to consider a ball because we can cover U will balls. The radius $1/2$ is not important; we could choose any larger number which is < 1.



1.3 Proof of the de Giorgi-Nash-Moser theorem

1.3.1 L^2 to L^∞ bound via Moser iteration

Step 1 of the proof is an L^2 to L^∞ bound.

Proposition 1.1. *Suppose that $Pw \leq 0$ and $w > 0$.*

$$\|w\|_{L^\infty} \lesssim_{d,\lambda,\Lambda} \|w\|_{L^2(B_1)}.$$

These conditions tells us that we cannot have a large peak to contribute to the L^∞ norm without contributing much to the L^2 norm.

Proof. (Moser iteration) Here are the ingredients:

1. **Lemma 1.1** (Energy estimate for $Pw \leq 0$, $w > 0$)**.** *For all* $\theta \in (0,1)$,

$$\|Dw\|_{L^2(B_{\theta R})} \lesssim \frac{\Lambda}{\lambda}\frac{1}{\theta R}\|w\|_{L^2(B_R)}.$$

Proof. Multiply by a cutoff χ which is 1 in $B_{\theta R}$ and 0 outside B_R and with $|D^\alpha| \lesssim \frac{1}{(\theta R)^{|\alpha|}}$. Then we use the energy method:

$$\begin{aligned} 0 &\geq \int Pw\chi^2 w\,dx \\ &= \int -\partial_k(a^{j,k}\partial_k w)\chi^2 w\,dx \\ &= \int a^{j,k}\partial_k w\chi^2\partial_j w\,dx + 2\int a^{j,k}\partial_j w\chi\partial_k\chi w. \end{aligned}$$

This means that

$$\begin{aligned} \int \chi^2 Dw^2\,dx &\leq \frac{1}{\lambda}\int a^{j,k}\partial_j w\partial_k w\chi^2\,dx \\ &\leq -2\frac{1}{\lambda}\int a^{j,k}\partial_j w\chi w\partial_k\chi \\ &\leq 2\frac{\Lambda}{\lambda}\int \chi|Dw||Dx||w|\,dx \\ &\lesssim \frac{1}{\theta R}\frac{\Lambda}{\lambda}\left(\int x^2|Dw|^2\,dx\right)^{1/2}\left(\int_{B_R}|w|^2\,dx\right)^{1/2}. \end{aligned}$$

Now cancel on both sides to get the result. □

2. Sobolev embedding: For $d \geq 3$, let $p* = \frac{2d}{d-2}$. If $\theta R < 1$,

$$\|w\|_{H^1(B_{\theta R})} \lesssim \frac{\Lambda}{\lambda}\frac{1}{\theta R}\|w\|_{L^2(B_R)}.$$

By the Sobolev ineuqality, we get a better L^p bound:

$$\|w\|_{L^{p*}(B_{\theta R})} \lesssim \frac{\Lambda}{\lambda}\frac{1}{\theta R}\|w\|_{L^2(B_R)}.$$

How do we iterate Step 2? The observation of Moser was that if $\beta > 1$, $Pw \leq 0$, and $w > 0$, then w^β satisfies $Pw \leq 0$ and $w > 0$; this is because the map $s \mapsto s^\beta$. Composing convex functions preserves convexity, and composing subsolutions gives a subsolution, as well. Therefore, we can apply 2 to w^β, we get

$$\|w\|^\beta_{L^{p_*\beta}(B_{\theta R})} \lesssim \frac{\Lambda}{\lambda}\frac{1}{\theta R}\|w\|^\beta_{L^{2\beta}(B_R)}.$$

We can rewrite this as

$$\|w\|_{L^{p_*\beta}(B_{\theta R})} \lesssim \left(\frac{\Lambda}{\lambda}\frac{1}{\theta R}\right)^{1/\beta} \|w\|^\beta_{L^{2\beta}(B_R)}.$$

If we denote $q = 2\beta$ and $\alpha = \frac{p_*}{2} > 1$, then this equation looks like

$$\|w\|_{L^{\alpha q}(B_{\theta R})} \lesssim \left(\frac{\Lambda}{\lambda}\frac{1}{\theta R}\right)^{2/q} \|w\|^\beta_{L^{q}(B_R)}.$$

We want to iterate this equation $(2q)$. Start with $q_0 = 2$, then apply this to $q_1 = 2\alpha$ and so on, so $q_n = 2\alpha^n$. What should our θs be so that the radius of the ball does not go to 0? The radii are $R_0 = 1$, $R_1 = \theta_1$, $R_2 = \theta_1\theta_2$, and so on, so $R_n = \theta_1 \cdots \theta_n$. The constants we get will be

$$C_1 = \left(\frac{\Lambda}{\lambda}\frac{1}{\theta_1 R_0}\right)^{2/q_0}, \cdots,$$

$$C_n = \left(\frac{\Lambda}{\lambda}\frac{1}{\theta_n R_{n-1}}\right)^{2/q_{n-1}} C_{n-1}\cdots C_1 = \left(\frac{\Lambda}{\lambda}\frac{1}{\theta_n \cdots \theta_1}\right)^{1/\alpha^{n-1}} \cdots \left(\frac{\Lambda}{\lambda}\frac{1}{\theta_{n-1} \cdots \theta_1}\right)^{1/\alpha^{n-2}}.$$

Our goal is to choose $\theta_1, \theta_2, \dots$ so that $\theta_1\theta_2\cdots = R_\infty = 1/2$. So we want

$$\theta_n^{-\frac{1}{\alpha^{n-1}}}\theta_{n-1}^{-\frac{1}{\alpha^{n-1}}-\frac{1}{\alpha^{n-2}}}\cdots\theta_1^{-\frac{1}{\alpha^{n-1}}\cdots-\frac{1}{\alpha}} = C_\infty < \infty.$$

If we let $a_n = \log\theta_n$, then we want $\exp(-\sum a_n) = 1/2$ and

$$\exp\left(\frac{1}{\alpha}a_1 + \frac{1}{\alpha^2}(a_1 + a_2) + \cdots + \frac{1}{\alpha^{n-1}}(a_1 + \cdots + a_{n-1}) + \cdots\right) < \infty. \qquad \square$$

These ingredients are the same things that de Giorgi's proof used, but his argument used sub-level sets instead of this iteration, so it was much more geometric.

1.3.2 Hölder seminorm bound via the de Giorgi oscillation lemma

The remaining step of the proof of the de Giorgi-Nash-Moser theorem is the following.

Proposition 1.2. *Let $w \in H^1(B_1)$ satisfy $Pw = 0$. Then there exists an $\alpha > 0$ such that*

$$[w]_{C^{0,\alpha}(B_{1/4})} \lesssim_{d,\lambda,\Lambda} \|w\|_{L^2(B_1)}.$$

This uses an oscillation lemma.

Lemma 1.2 (de Giorgi oscillation lemma)**.** *There exists a $\gamma \in (0,1)$ such that for $w \in H^1(B_1)$ with $Pw = 0$,*

$$\operatorname{osc}_{B_{1/2}} w \leq \gamma \operatorname{osc}_{B_1} w,$$

where $\operatorname{osc}_U w := \sup_U w - \inf_U w$.

Here is how the lemma implies the proposition.

Proof. The idea is to let $D = |x-y|$ and apply the oscillation lemma iteratively to get

$$|w(x) - w(y)| \leq \operatorname{osc} B_{B_D(w)} w \leq \gamma^n \operatorname{osc}_{B_{2^n D(x)}} w$$

Now let $n = -\log_2 D + c$ so that $B_{2^n D} \subseteq B_1$. We get

$$\begin{aligned} &\lesssim \gamma^n \|w\|_{L^\infty(B_1)} \\ &\lesssim D^\alpha \|w\|_{L^2(B_2)} \end{aligned}$$

where $\alpha = -\log_2 \gamma > 0$, so $\gamma^n = \gamma^{-\log_2 D}$. $\square$

1.3.3 The de Giorgi-Harnack inequality

The way to prove the de Giorgi oscillation lemma lemma is to see that w should satisfy a sort of Harnack inequality.

Lemma 1.3 (de Giorgi-Harnack inequality)**.** *Let $w \in H^1(B_1)$ with $1 > w > 0$ and $Lw = 0$. Assume that*

$$\left|\left\{x \in B_{1/2} : w \geq \frac{1}{2}\right\}\right| \geq \frac{1}{2}|B_{1/2}|.$$

Then there exists a $\gamma > 0$ such that $w \geq \gamma$ in $B_{1/2}$.

Here is how the de Giorgi-Harnack inequality implies the oscillation lemma.

Proof. Without loss of generality, we may arrange for $\sup_{B_1} w = 1 - \varepsilon$ and $\inf_{B_1} w = \varepsilon$. On $B_{1/2}$, one of the following must hold:

1. $|\{x \in B_{1/2} : w \geq \frac{1}{2}\}| \geq \frac{1}{2}|B_{1/2}|$: In this case, apply the de Giorgi-Harnack inequality for w.

2. $|\{x \in B_{1/2} : w \geq \frac{1}{2}\}| \leq \frac{1}{2}|B_{1/2}|$: This this case, $1 - w$ is still a solution, so we can apply he de Giorgi-Harnack inequality for $1 - w$. $\square$

Moser's approach actually proves the de Giorgi-Harnack inequality without the last assumption, but this needs PMO theory. Here is a quick proof of the inequality:

Proof. The key idea is to look at $v = -\log w$. (Exercise: For $-\Delta u = 0$ in U, show that $|\nabla \log u|_{L^\infty(V)} \lesssim 1$ for all $V \subseteq\subseteq U$. Then get that $\min_V u \geq \gamma \max_V u$.) There is an a priori bound for $\nabla \log w$:

Lemma 1.4. *Suppose $w \in H^1(B_1)$ with $Pw \geq 0$ and $w > 0$. Then*

$$\|\nabla \log w\|_{L^2(B_{1/2})} \lesssim \frac{\Lambda}{\lambda}.$$

Proof. Multiply $Pw \geq 0$ by w^{-1} and integrate over U. □

This is deficient in two ways: it is not an L^∞ bound, and it is only a bound on the gradient, not w itself. However, notice that w is also a subsolution, so $v = -\log w$ is a subsolution: $Pv \geq 0$. When $w < 1$, $v > 0$. So we have inequality of the form

$$\|v\|_{L^\infty(B_{1/4})} \lesssim \|v\|_{L^2(B_{1/2})}.$$

The last assumption in the statement of the de Giorgi-Harnack inequality tells us that

$$|\{x \in B_{1/2} : v \leq \log 2\}| \geq \frac{1}{2}|B_{1/2}|.$$

Now we use a Poincaré-type inequality:

Lemma 1.5. *If the above bound (*) holds, ad $v \in H^1(B_{1/2})$, then*

$$\|v\|_{L^2(B_{1/2})} \lesssim \|Dv\|_{L^2(B_{1/2})} + 1.$$

Proof. By the standard Poincaré inequality, there exists a c such that

$$\|v - c\|_{L^2(B_{1/2})} \lesssim \|Dv\|_{L^2(B_{1/2})}.$$

Now split into cases: If $c \leq 100 \log 2$, we are done. If $c \geq 100 \log 2$, then

$$\begin{aligned} \|Dv\|_{L^2(B_{1/2})} \geq \|v - c\|_{L^2(B_{1/2})} & \\ &\geq \|v - c\|_{L^2(A)} \\ &\geq \frac{99}{100} c \|1\|_{L^2(A)} \\ &\gtrsim c, \end{aligned}$$

where the last step uses the above bound (*). □

This completes the proof of the Giorgi-Harnack inequality. □